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NOTE

Another Simple Proof of a Theorem of Milner

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An *intersecting Sperner system* on $[n]=\{1, ..., n\}$ is a collection of subsets of $[n]$, no pair of which is either disjoint or nested. Milner [2] proved that an intersecting Sperner system on $[n]$ has at most

$$\binom{n}{\lceil (n+1)/2 \rceil}$$

sets. Katona [1] gave a simple proof of Milner's theorem using the cycle method. We give a simpler proof that uses the cycle method in a different way.

We write $[n]^{(k)}$ for the set of subsets of size k of $[n]$. For $\mathscr{F} \subset [n]^{(k)}$ we write $\partial^{+}\mathscr{F}$ for the upper shadow $\{G \in [n]^{(k+1)} : G \supset F \text{ for some } F \in \mathscr{F}\}$ of $\mathscr{F}$ and $\partial^{-}\mathscr{F}$ for the lower shadow $\{G \in [n]^{(k-1)} : G \subset F \text{ for some } F \in \mathscr{F}\}$. By a simple counting argument, if $k<n/2$ then $|\partial^{+}\mathscr{F}| \geqslant |\mathscr{F}|$ and if $k>n/2$ then $|\partial^{-}\mathscr{F}| \geqslant |\mathscr{F}|$.

THEOREM 1. *An intersecting Sperner system on $[n]$ has size at most*

$$\binom{n}{\left\lceil \frac{n+1}{2} \right\rceil}. \tag{1}$$

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Proof. Let $\mathscr{F} \subset \mathscr{P}(n)$ be an intersecting Sperner system of maximum size N. If n is odd, then $\mathscr{F}$ satisfies (1) by Sperner's lemma, so we may assume $n = 2k$ is even. Let $r = \min\{|A| : A \in \mathscr{F}\}$ and, for $0 \leqslant k \leqslant n$, $\mathscr{F}_k = \mathscr{F} \cap [n]^{(k)}$. If $r < n/2 = k$ then consider the system $\mathscr{F}' = (\mathscr{F} \setminus \mathscr{F}_r) \cup \partial^+ \mathscr{F}_r$. This is an intersecting Sperner system which is at least as large as $\mathscr{F}$, since $|\partial^+ \mathscr{F}_r| \geqslant |\mathscr{F}_r|$. Repeating the argument, we may assume that $|A| \geqslant n/2$ for $A \in \mathscr{F}$. Now let $r = \max\{|A|: A \in \mathscr{F}\}$. If $r > k+1$ then consider $\mathscr{F}' = (\mathscr{F} \setminus \mathscr{F}_r) \cup \partial^- \mathscr{F}_r$. Since all sets in $\mathscr{F}$ have size at least $n/2$, this is an intersecting Sperner system, and $|\mathscr{F}'| \geqslant |\mathscr{F}|$ because $|\partial^- \mathscr{F}_r| \geqslant |\mathscr{F}_r|$. Repeating, we may assume that $\mathscr{F} \subset [n]^{(k)} \cup [n]^{(k+1)}$.

Let $\mathscr{G} = \partial^+ \mathscr{F}_k$. Since $\mathscr{G}$ and $\mathscr{F}_{k+1}$ are disjoint and $|\mathscr{F}_{k+1}| + |\mathscr{G}|$ is bounded by (1), the theorem follows if we show that $|\mathscr{G}| \geqslant |\mathscr{F}_k|$.

Consider a cyclic order $\mathbf{c}$ of $[n]$ and suppose $f(\mathbf{c})$ elements of $\mathscr{F}_k$ and $g(\mathbf{c})$ elements of $\mathscr{G}$ occur as intervals in $\mathbf{c}$. Since we do not have both an interval and its complement in $\mathscr{F}_k$, we have $f(\mathbf{c}) \leqslant n/2 = k$. However, every interval of length k can be extended to an interval of length $k+1$ in two ways, so $g(\mathbf{c}) \geqslant f(\mathbf{c}) + 1 \geqslant ((k+1)/k)\, f(\mathbf{c})$. Each element of $\mathscr{F}_k$ occurs in $k!^2$ cyclic orders and each element of $\mathscr{G}$ in $(k+1)!\,(k-1)!$ cyclic orders, so summing over all orders gives

$$(k+1)!\,(k-1)!\,|\mathscr{G}| = \sum_{\mathbf{c}} g(\mathbf{c}) \geqslant \frac{k+1}{k} \sum_{\mathbf{c}} f(\mathbf{c}) = \frac{k+1}{k} k!^2\,|\mathscr{F}_k|,$$

and so $|\mathscr{F}_k| \leqslant |\mathscr{G}|$, as required. ∎

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